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ON QUANTIZED ALGEBRA OF WESS-ZUMINO DIFFERENTIAL OPERATORS AT ROOTS OF UNITY

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1. Motivation. In [13] J. Wess and B. Zumino constructed a broad family of quantum deformations of the algebra of scalar differential operators on the affine space. If the deformation parameter q is generic, this algebra has the same dimension as in non-deformed case; if q is a root of unity, then the dimension of deformed algebra is less than that of non-deformed algebra. However, this does not quite agree with the intuition gained in the classical theory of differential operators. In fact, due to the analogy between quantum case at roots of unity and the classical case in positive characteristic, one should expect the appearance of differential operators that are not compositions of operators of order $\leqslant 1$, whereas the dimension of the whole algebra must be the same as in non-deformed case. In this short paper it is shown that this intuition is a true one: we quantize the standard algebraic definition of differential operator $[\ldots[[D, f_0], f_1]\ldots, f_k] = 0$ by replacing the usual commutators by twisted ones and obtain the algebra of differential operators that has the classical dimension for any q and coincides with the Wess-Zumino algebra at generic parameter values. A detailed description of this algebra is presented. Note also that having such quantum differential operators one can construct quantum jets, de Rham and Spencer complexes, integral forms, Euler operator, and so on, following the same logic as in non-deformed case (see [9, 10, 12, 11, 7]). A realization of this program has been started in [8].

2. Here we recall some standard facts about quantum affine spaces. We refer to [6, 2, 1, 4] for further details.

Let $\Bbbk$ be a commutative ground ring with unity. Consider a free $\Bbbk$-module V and a non-degenerate linear operator $\widehat{R}\colon V \otimes V \to V \otimes V$. Throughout this paper we assume that the operator $\widehat{R}$ satisfies the Yang-Baxter equation

$$\widehat{R}_{12}\widehat{R}_{23}\widehat{R}_{12} = \widehat{R}_{23}\widehat{R}_{12}\widehat{R}_{23}$$

(here $\widehat{R}_{12} = \widehat{R} \otimes 1$, $\widehat{R}_{23} = 1 \otimes \widehat{R}$) and the Hecke condition

$$\widehat{R}^2 = \left(q - q^{-1}\right)\widehat{R} + 1$$

for some invertible element q of $\Bbbk$.

Basic example ([3]). Let $\{\xi_1, \dots, \xi_n\}$ be a basis of V and $\{x^1, \dots, x^n\}$ be the dual basis of V^*. Let $\|\widehat{R}^{ij}_{kl}\|$ be the matrix of $\widehat{R}$ in this basis. Choose an invertible $q \in \Bbbk$ and put

$$\widehat{R}^{ij}_{kl} = \delta^i_l \delta^j_k \left(1 + (q-1)\,\delta^{ij}\right) + \left(q - q^{-1}\right) \delta^i_k \delta^j_l \theta\,(j-i)\,, \tag{1}$$

where $\theta\,(i) = \begin{cases} 1, & \text{for } i > 0 \\ 0, & \text{for } i \leqslant 0. \end{cases}$

The algebra A of a *quantum affine space* is defined as the quotient algebra of the tensor algebra $T\,(V^*)$ by the ideal generated by the image of $q - \widehat{R}^* \colon V^* \otimes V^* \to V^* \otimes V^*$. In coordinates, A is generated by $x^1, \dots, x^n$ subject to the relations

$$\widehat{R}^{ij}_{\alpha\beta} x^\alpha x^\beta = q x^i x^j$$

(we assume summation over repeated indices which occur in both upper and lower positions).

Example. Assume that $\widehat{R}$ is given by (1). Then A is generated by $x^1, \dots, x^n$ modulo the following relations:

$$x^i x^j = q x^j x^i, \quad i < j.$$

Consider the matrix algebra M defined as the quotient of the tensor algebra $T\,(\mathrm{End}\,(V \otimes V))$ by the ideal generated by the elements $\widehat{R}F - F\widehat{R}$, where $F \in \mathrm{End}\,(V \otimes V)$. The algebra M is generated by t^i_j, $1 \leqslant i, j \leqslant n$, obeying the relations

$$\widehat{R}^{ij}_{\alpha\beta} t^\alpha_k t^\beta_l = t^i_\alpha t^j_\beta \widehat{R}^{\alpha\beta}_{kl}.$$

There is a natural bialgebra structure on M, with the comultiplication given by $\Delta t^i_j = t^i_\alpha \otimes t^\alpha_j$ and the counit $\varepsilon t^i_j = \delta^i_j$. One obtains a matrix *quantum group* H by taking the Hopf envelop of M, that is an initial object in a category of bialgebra morphisms $M \to C$, with C being a Hopf algebra (see [5]).

The algebra A is a (right) comodule-algebra over H, with the coaction given by

$$\Delta\left(x^i\right) = x^\alpha \otimes \tilde{t}^i_\alpha,$$

where the tilde stands for the antipode.

Remark. The antipode was inserted here to make this transformation *right* coaction.

We shall assume that the Hopf envelop H of the matrix bialgebra is endowed with a cobraided structure, i.e., that there exists a non-degenerate bilinear form $\langle \cdot\,, \cdot \rangle$ on H satisfying the conditions:

$$\langle h, h_1 h_2\rangle = \langle h_i, h_2\rangle \left\langle h^i, h_1\right\rangle$$
$$\langle h_1 h_2, h\rangle = \langle h_1, h_i\rangle \left\langle h_2, h^i\right\rangle$$
$$g_j h_i \left\langle h^i, g^j\right\rangle = \langle h_i, g_j\rangle\, h^i g^j$$

for all h, $h_1, h_2, g \in H$ and $\Delta h = h_i \otimes h^i$, $\Delta g = g_j \otimes g^j$. On the generators this form will be given by

$$\left\langle t^i_j, t^k_l\right\rangle = q(\widehat{R}^{-1})^{ki}_{jl}.$$

It is well-known that such a structure does exist in all examples one is likely to encounter, provided H was constructed in an appropriate category.

Having this form we can introduce a (left) H-module structure on the algebra A:

$$h \cdot f = f_i \left\langle h, h^i\right\rangle,$$

where $h \in H$, $f \in A$, $\Delta f = f_i \otimes h^i$. In coordinates we have

$$t^i_j x^k = C^{ik}_{\alpha j} x^\alpha, \quad t^i_j \cdot 1 = \delta^i_j,$$
$$\tilde{t}^i_j x^k = q(\widehat{R}^{-1})^{ki}_{j\alpha} x^\alpha,$$
$$\tilde{\tilde{t}}^i_j x^k = q^{-1} \widehat{R}^{ik}_{\alpha j} x^\alpha,$$

where the matrix C is defined by $C^{i\alpha}_{j\beta}(\widehat{R}^{-1})^{k\beta}_{l\alpha} = q^{-1}\delta^i_l \delta^k_j$. Note that this makes A a crossed H-bimodule.

Lemma. *For any $f \in A$ we have*

$$x^i f = \left(\tilde{\tilde{t}}^i_\alpha f\right) x^\alpha$$
$$f x^i = x^\alpha \left(\tilde{t}^i_\alpha f\right).$$

Proof. We prove the first relation, the second can be proved in the same way. For $f = 1$ the relation is obvious. Suppose inductively that it is true for all f such that $\deg f < m$. Take $f = x^j g$ with $\deg g < m$. Then we get

$$\left(\tilde{\tilde{t}}^i_\alpha f\right) x^\alpha = \left(\tilde{\tilde{t}}^i_\beta x^j\right)\left(\tilde{\tilde{t}}^\beta_\alpha g\right) x^\alpha = \left(\tilde{\tilde{t}}^i_\beta x^j\right) x^\beta g = q^{-1}\widehat{R}^{ij}_{\alpha\beta} x^\alpha x^\beta g = x^i x^j g = x^i f.$$

Therefore the relation holds true for any$f$ of degree m. □

Having an H-module structure on A we can define an H-module structure on $\mathrm{Hom}_{\Bbbk}(A, A)$ by the formula:

$$\left(t^i_j D\right)(f) = t^i_\alpha\left(D\left(\tilde{t}^\alpha_j f\right)\right), \quad D \in \mathrm{Hom}_{\Bbbk}(A, A).$$

Notice that $\mathrm{Hom}_{\Bbbk}(A, A)$ is a module-algebra over H:

$$\left(t^i_j\left(D_1 \circ D_2\right)\right) = t^i_\alpha D_1 D_2 \tilde{t}^\alpha_j = t^i_\beta D_1 \tilde{t}^\beta_\varepsilon t^\varepsilon_\alpha D_2 t^\alpha_j = \left(t^i_\varepsilon D_1\right) \circ \left(t^\varepsilon_j D_2\right).$$

3. Now we define quantum algebra of (scalar) differential operators. For any $\Bbbk$-linear mapping $D\colon A \to A$ and for all $k \geqslant 0$, $1 \leqslant i \leqslant n$, define the mappings $\left[D, x^i\right]_k : A \to A$ by the formula:

$$\left[D, x^i\right]_k(f) = D\left(x^i f\right) - q^{2k} x^\alpha\left(\tilde{t}^i_\alpha D\right)(f).$$

The set of differential operators $\mathrm{Diff}_k(A)$ of order $\leqslant k$ can be defined by the following inductive procedure: $\mathrm{Diff}_0(A) = \{D \in \mathrm{Hom}_{\Bbbk}(A, A) \mid \left[D, x^i\right]_0 = 0\ \forall i\}$. In general, $\mathrm{Diff}_{k+1}(A) = \{D \in \mathrm{Hom}_{\Bbbk}(A, A) \mid \left[D, x^i\right]_{k+1} \in \mathrm{Diff}_k(A)\ \forall i\}$.

Proposition. (1) *If* $D \in \mathrm{Diff}_k(A)$ *then* $h \cdot D \in \mathrm{Diff}_k(A) \quad \forall h \in H$.

(2) *If* $D_1 \in \mathrm{Diff}_k(A)$ *and* $D_2 \in \mathrm{Diff}_l(A)$, *then* $D_1 \circ D_2 \in \mathrm{Diff}_{k+l}(A)$.

Proof. (1) It is sufficient to prove that

$$t^j_l\left[D, x^i\right]_k = C^{\alpha i}_{\beta l}\left[t^j_\alpha D, x^\beta\right]_k.$$

We have

$$
\begin{aligned}
t^j_l \left[D, x^i\right]_k &= \left(t^j_\alpha D\right)\left(t^\alpha_l x^i\right) - q^{2k}\left(t^j_\beta x^\alpha\right)\left(t^\beta_l \tilde{t}^i_\alpha D\right) \\
&= C^{\alpha i}_{\beta l}\left(t^j_\alpha D\right) x^\beta - q^{2k} C^{j\alpha}_{\varepsilon\beta} x^\varepsilon \left(t^\beta_l \tilde{t}^i_\alpha D\right) \\
&= C^{\alpha i}_{\beta l}\left(t^j_\alpha D\right) x^\beta - q^{2k} C^{\alpha i}_{\beta l} x^\varepsilon \left(\tilde{t}^\beta_\varepsilon t^j_\alpha D\right) \\
&= C^{\alpha i}_{\beta l}\left[t^j_\alpha D, x^\beta\right]_k .
\end{aligned}
$$

(2) $\left[D_1 \circ D_2, x^i\right]_{k+l}$

$$
\begin{aligned}
&= D_1 D_2 x^i - q^{2l} D_1 x^\alpha \left(\tilde{t}^i_\alpha D_2\right) + q^{2l} D_1 x^\alpha \left(\tilde{t}^i_\alpha D_2\right) - q^{2k+2l} x^\beta \left(\tilde{t}^\alpha_\beta D_1\right)\left(\tilde{t}^i_\alpha D_2\right) \\
&\qquad = D_1 \circ \left[D_2, x^i\right]_l + q^{2l}\left[D_1, x^\alpha\right]_k \circ \tilde{t}^i_\alpha D_2
\end{aligned}
$$

This implies the statement in a standard way. □

This Proposition has the following:

Corollary. (1) *There exists an A-bimodule structure on* $\operatorname{Diff}_k(A)$:

$$a \cdot D = a \circ D, \quad D \cdot a = D \circ a.$$

(2) *For any $k < l$ we have* $\operatorname{Diff}_k(A) \subset \operatorname{Diff}_l(A)$.

We denote $\operatorname{Diff}(A) = \bigcup_{k \geqslant 0} \operatorname{Diff}_k(A)$.

4. Now we give a detailed description of the modules $\operatorname{Diff}_k(A)$. We shall assume that the space V^* has an ordered basis $x^1, \dots, x^n$ such that the monomials $\left(x^1\right)^{i_1} \dots \left(x^n\right)^{i_n}$ form a basis of A and $\widehat{R}^{ij}_{kl} = 0$ if $i > j$ and either $k > l$ or $k \geqslant i$.

Clearly, any differential operator $D \in \operatorname{Diff}_k(A)$ is completely determined by its values on the basic monomials of degree $\leqslant k$. The following theorem states that these values can be taken in an arbitrary way.

Theorem. *The A-module* $\operatorname{Diff}_k(A)$ *has a basis* 1, ∂_σ, $|\sigma| \leqslant k$, *where* $\sigma = j_1 \dots j_r$, $1 \leqslant j_l \leqslant j_{l+1} \leqslant n$, *is a multi-index,* $|\sigma| = r$, *and the operator* ∂_σ *is equal to* 1 *at* $x^{j_1} \dots x^{j_r}$ *and vanishes on the other monomials of degree* $\leqslant k$.

Proof. We have to show that the defining relations $\left[\cdots\left[D, x^{i_k}\right]_k \cdots x^{i_0}\right]_0 = 0$ with $i_{l+1} \leqslant i_l$ will be sufficient for $D \in \mathrm{Diff}_k(A)$. For this we verify that

$$\left[\left[D, x^i\right]_{l+1}, x^j\right]_l = q^{-1}\widehat{R}^{ij}_{\alpha\beta}\left[\left[D, x^\alpha\right]_{l+1}, x^\beta\right]_l .$$

We have

$$\begin{aligned}
&\left[\left[D, x^i\right]_{l+1}, x^j\right]_l \\
&= Dx^i x^j - q^{2l+2}x^\alpha\left(\tilde{t}^i_\alpha D\right)x^j - q^{2l}x^\alpha\left(\tilde{t}^\gamma_\alpha D\right)\left(\tilde{t}^j_\gamma x^i\right) + q^{4l+2}x^\alpha\left(\tilde{t}^\gamma_\alpha x^\varepsilon\right)\left(\tilde{t}^j_\gamma \tilde{t}^i_\varepsilon D\right) \\
&= Dx^i x^j - q^{2l+1}\left(q\delta^i_\gamma\delta^j_\varepsilon + (\widehat{R}^{-1})^{ij}_{\gamma\varepsilon}\right)\left(x^\alpha\left(\tilde{t}^\gamma_\alpha D\right)x^\varepsilon\right) + q^{4l+2}x^\varepsilon x^\gamma\left(\tilde{t}^j_\gamma \tilde{t}^i_\varepsilon D\right) \\
&= q^{-1}\widehat{R}^{ij}_{\alpha\beta}Dx^\alpha x^\beta - q^{2l}\widehat{R}^{ij}_{\alpha\beta}\left(q\delta^\alpha_\gamma\delta^\beta_\varepsilon + (\widehat{R}^{-1})^{\alpha\beta}_{\gamma\varepsilon}\right)\left(x^\varkappa\left(\tilde{t}^\gamma_\varkappa D\right)x^\varepsilon\right) \\
&\qquad + q^{4l+1}\widehat{R}^{ij}_{\alpha\beta}x^\varepsilon x^\gamma\left(\tilde{t}^\beta_\gamma \tilde{t}^\alpha_\varepsilon D\right) = q^{-1}\widehat{R}^{ij}_{\alpha\beta}\left[\left[D, x^\alpha\right]_{l+1}, x^\beta\right]_l . \quad \square
\end{aligned}$$

5. We can define a (right) H-comodule structure on $\mathrm{Diff}_k(A)$ by the following property:

$$h \cdot D = D_i\left\langle h, h^i\right\rangle,$$

where $h \in H$, $D \in \mathrm{Diff}_k(A)$, $\Delta D = D_i \otimes h^i$. In particular, for the operators ∂_i this yields:

$$\Delta\partial_i = \partial_\alpha \otimes t^\alpha_i .$$

This structure makes $\mathrm{Diff}(A)$ an H-comodule-algebra:

$$\begin{aligned}
h \cdot (D_1 \circ D_2) = (h_i D_1)\left(h^i D_2\right) &= (D_1)_k (D_2)_l \left\langle h_i, g^k_1\right\rangle\left\langle h^i, g^l_2\right\rangle \\
&= (D_1)_k (D_2)_l \left\langle h, g^l_2 g^k_1\right\rangle,
\end{aligned}$$

so that

$$\Delta(D_1 \circ D_2) = (D_1)_k \circ (D_2)_l \otimes g^l_2 g^k_1,$$

where $\Delta h = h_i \otimes h^i$, $\Delta D_r = (D_r)_k \otimes g^k_r$, $r = 1, 2$.

Proposition (Covariance of differential operators). *For any $D \in \mathrm{Diff}_k(A)$ and $f \in A$*

$$\Delta(D(f)) = \Delta(D)(\Delta(f)),$$

where the action in right-hand side is given by $(\nabla \otimes h_1)(a \otimes h_2) = \nabla(a) \otimes h_2 h_1$, $\nabla \in \mathrm{Diff}_k(A)$, $a \in A$, $h_1, h_2 \in H$.

Proof. For f of degree zero the statement is trivial. Suppose inductively that it is true for $\deg f \leqslant m$ and take an element f of degree m. We get

$$\begin{aligned}\Delta\left(D\left(x^i f\right)\right) = \Delta\left(\left(Dx^i\right)(f)\right) = \Delta\left(Dx^i\right)(\Delta(f)) = \Delta(D)\,\Delta\left(x^i\right)\Delta(f)\\ = \Delta(D)\left(\Delta\left(x^i f\right)\right). \quad \square\end{aligned}$$

6. Consider the first order operators ∂_i. It follows easily from the very definition that they satisfy the Wess-Zumino Leibnitz rule:

$$\partial_i x^j = \delta_i^j + q\widehat{R}_{i\alpha}^{j\beta} x^\alpha \partial_\beta.$$

Further, one has the following commutation relations between ∂_i:

$$\widehat{R}_{ji}^{\beta\alpha}\partial_\alpha\partial_\beta = q\partial_i\partial_j.$$

Indeed, it is obvious that the second order operator $\widehat{R}_{ji}^{\beta\alpha}\partial_\alpha\partial_\beta - q\partial_i\partial_j$ vanishes on the monomials of degree 0 and 1. For the degree 2 we have:

$$\begin{aligned}\left(\widehat{R}_{ji}^{\beta\alpha}\partial_\alpha\partial_\beta - q\partial_i\partial_j\right)\left(x^k x^l\right) = \widehat{R}_{ji}^{kl} + q\widehat{R}_{ji}^{\beta\alpha}\widehat{R}_{\beta\alpha}^{kl} - q\delta_i^l\delta_j^k - q^2\widehat{R}_{ji}^{kl}\\ = q(\widehat{R}^2 - \left(q - q^{-1}\right)\widehat{R} - 1)_{ji}^{kl} = 0.\end{aligned}$$

Definition ([2]). The *quantum Weyl algebra* $\mathcal{D}$ is the algebra with $2n$ generators $x^1, \dots, x^n, \partial_1, \dots, \partial_n$ satisfying the following commutation relations:

$$\begin{aligned}\widehat{R}_{\alpha\beta}^{ij} x^\alpha x^\beta &= q x^i x^j,\\ \widehat{R}_{ji}^{\beta\alpha}\partial_\alpha\partial_\beta &= q\partial_i\partial_j,\\ \partial_i x^j &= \delta_i^j + q\widehat{R}_{i\alpha}^{j\beta} x^\alpha\partial_\beta.\end{aligned}$$

From the previous discussion it follows that there exists an algebra morphism $w\colon \mathcal{D} \to \mathrm{Diff}(A)$ such that $w\left(x^i\right) = x^i$, $w(\partial_j) = \partial_j$. The image of w consists of differential operators that are compositions of operators of order $\leqslant 1$. One can also show that $\operatorname{coker} w$ is exactly the zero Spencer cohomology of the algebra A (see [12]). For the matrix $\widehat{R}$ of the form (1) from the above-stated Theorem and a result of E. Demidov [2, Corollary 12.6] it follows the following:

Proposition. *If q^2 is not a root of unity, or $q^2 = 1$ and* $\operatorname{char}\Bbbk = 0$, *then* $w\colon \mathcal{D} \to \operatorname{Diff}(A)$ *is an isomorphism, and if q^2 is a primitive root of unity of degree ℓ, then the kernel of w is generated by $\partial_1^\ell, \dots, \partial_n^\ell$.*

Thus, in the root of unity case there exist operators that can not be presented as a composition of operators of order $\leqslant 1$.

Example. Consider the operators $D_j = \partial_{\underbrace{j\dots j}_{\ell \text{ times}}}$. It is easy to see that $\partial_i\left(\left(x^j\right)^\ell\right) = 0 \quad \forall i$. Hence D_j is not a composition of operators of order $\leqslant 1$. In fact, any operator is a composition of D_j and operators of order $\leqslant 1$.

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